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M.A./M.Sc. Semester III Examination, 2019 (Old pattern under CDOE)
Subject: Mathematics (Applied Stream)
Paper: MAG301 (Methods of Applied Mathematics-I)

Time: 2 Hours — Full Marks: 45

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
[Notation and symbols have their usual meaning]

Answer any **five** questions. Only **first five** answers will be evaluated. 9×5 = 45

1. (a) If $f(x)$ is piecewise continuously differentiable and absolutely integrable function, then prove that $F(k)$, Fourier transform of $f(x)$, is bounded and continuous in $0 < k < \infty$. [5]

   (b) Find the Fourier transform of $f(x) = x\exp(-a|x|), a > 0$. [4]

2. (a) Let $f(x)$ be a piecewise continuous function of exponential order α on $[0,\infty)$. Prove that Laplace transform of $\int_t^\infty \frac{f(x)}{x}dx$ is $\frac{1}{s}\int_0^s F(t)dt$, where $F(s)$ is the Laplace transform of $f(x)$. [3]

   (b) Use the Convolution theorem to find the inverse Laplace transforms of $F(s) = \frac{1}{s^2(s^2+a^2)}$. [6]

3. (a) Solve the integral equation, $y(x) = x + \lambda\int_0^{2\pi}|\pi - t|\sin x\; y(t)dt$. [4]

   (b) Solve the following integral equation, $\int_1^x \frac{y(t)dt}{(\cos t - \cos x)^{\frac{1}{2}}} = x, 1 < x < 2$. [5]

4. (a) Solve, $y(x) = \frac{5x}{6} - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)y(t)dt$ by the method of successive approximations. [5]

   (b) Let $k_m(x,t)$ be the iterated kernel of the integral equation $y(x) = f(x) + \lambda\int_a^b k(x,t)y(t)dt$. Prove that the series for resolvent kernel $R(x,t;\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t)$ is absolutely and uniformly convergent for all values of x and t in the circle $|\lambda| < \frac{1}{||K||}$, where $\| k \| = \left(\int_a^b\int_a^b |K(x,t)|^2 dx\, dt\right)^{\frac{1}{2}}$. [4]

5. (a) Prove that the adjoint operator A^* of a bounded operator A on a Hilbert space is bounded. Also, show that $\| A \| = \| A^* \|$ and $\| A^*A \| = \| A \|^2$. [3+2+1=6]

   (b) If A is a bounded self-adjoint operator on a Hilbert space, then prove that the spectral radius of A, $r(A) = \| A \|$. [3]

6. Using Hilbert-Schmidt theorem, find the solution of the integral equation, [9]

$$y(x) = 1 + \frac{3}{2}\int_0^\pi \cos(x+t)\; y(t)dt.$$

7. (a) Does pointwise convergence imply convergence in the sense of generalized function? Justify your answer. Is the converse true? Support your answer. [3+2]

   (b) Find the differentiation of $|x|$ in the sense of generalized function. [2]

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(c) Find the Fourier transform of Dirac delta function $\delta(x+a)$ in the sense of generalized function. [2]

M.A./M.Sc. Semester III Examination, 2019 (Old pattern under CDOE)
Subject: Mathematics (Pure Stream)
Paper: MPG301

Time: 2 Hours Full Marks: 45

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
[Notation and symbols have their usual meaning]

Group-A (Modern Algebra-II)
(Marks: 27)

Answer any **three** questions. Only **first three** answers will be evaluated. 9×3 = 27

1. (a) Giving a suitable example, show that the external direct product of two infinite cyclic groups is not cyclic. [4]

   (b) Let G be a group with an identity element e. Let H and K be two normal subgroups of G. Show that G is an internal direct product of H and K if and only if $G = HK$ and $H \cap K = \{e\}$. [5]

2. (a) Let p be a prime. Prove that every finite $p-$group is of order p^n for some $n > 0$. [5]

   (b) Determine all non-isomorphic abelian groups of order $5^2 7^3$. [4]

3. (a) Let H and K be two subgroups of a group G. Prove that the number of distinct conjugates of H induced by the elements of K is equal to $[K : N_K(H)]$, the index of $N_K(H)$ in K. [3]

   (b) Let G be a group of order 77 acting on a set S of 20 elements. Show that G must have a fixed point. [3]

   (c) Does there exist a simple group of order 34? Justify your answer. [3]

4. (a) Determine positive integer n such that $\mathbb{Z}_n$ has no non-zero nilpotent elements. [3]

   (b) Let a,b be two non-zero elements of a unique factorisation domain D. If $\gcd(a,b) = 1$ and $a \mid c$, $b \mid c$, then prove that $ab \mid c$ in D, where $c \in D$. [3]

   (c) Show that a polynomial ring $\mathbb{Z}[x]$ is a UFD, but not a PID. [3]

5. (a) Prove that if R is a Noetherian ring, then so is $R[x]$. [6]

   (b) Show that $f(x) = x^4 - 2x^2 + 8x + 1$ is irreducible in $\mathbb{Q}[x]$. [3]

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Group-B (General Topology-I)

(**Marks: 18**)

Answer any **two** questions. Only **first two** answers will be evaluated. 9×2 = 18

1. (a) Define a normal space. Is normality a hereditary property? Support your answer. [1+3]

   (b) Prove that a topological space is completely normal if and only if every subspace of it is normal. [5]

2. (a) Let $\{(X_\alpha,\ \tau_\alpha): \alpha \in \Lambda\}$, Λ being index set, be a family of topological spaces and (X, τ) be their product space. Prove that a sequence $\{s_n\}_{n\in\mathbb{N}}$ in (X, τ) converges to a point $x = (x_\alpha)_{\alpha\in\Lambda} \in X$ if and only if the sequence $\{p_\alpha(s_n)\}_{n\in\mathbb{N}}$ converges to x_α in the α-th co-ordinate space $(X_\alpha,\ \tau_\alpha)$ for each $\alpha \in \Lambda$, where $p_\alpha: X \to X_\alpha$ is the α-th projection map, $\alpha \in \Lambda$. [5]

   (b) Give an example with proper justification of a second countable Frechet compact space which is not countably compact. [4]

3. (a) Define a locally compact topological space. Let (X, τ) be a locally compact space, $(Y,\ \sigma)$ be any topological space and $f: (X,\ \tau) \to (Y,\ \sigma)$ be an open continuous surjection. Prove that $(Y,\ \sigma)$ is locally compact. [1+3]

   (b) Let $\{\, S_n,\ n \in D,\ \geq \}$ be a net in a topological space (X, τ). Prove that there exists a filter $\mathfrak{F}$ on X such that if $x \in X$ is a cluster point of the net $\{\, S_n,\ n \in D,\ \geq \}$, then x is also a cluster point of the filter $\mathfrak{F}$ and vice-versa. [5]