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M.A./M.Sc. Semester I Examination, 2019 (Old pattern under CDOE)

Subject: Mathematics

Paper: MCG101

Time: 2 Hour Full Marks: 45

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning.]

Write the answer to Questions of each Group in separate books.

Group–A (Functional Analysis)

[Marks: 27]

Answer any **three** questions. Only **first three** answers will be evaluated. $9\times 3 = 27$

1. (a) Give an example with proper justifications of an incomplete normed linear space. [5]

 (b) Let us consider the metric space (X, d), where $X = \left(0,\ \frac{1}{6}\right)$ $(\subset\ \mathbb{R})$ and d is the metric induced on X by the usual metric of $\mathbb{R}$. Let us define $T:(X,d)\ \rightarrow\ (X,d)$ by $T(x) = x^2,\ x \in X$. Is T a contraction map? Justify your answer. Does there exist any fixed point of T? Support your answer. [2+2]

2. (a) If a normed linear space X is finite dimensional, then prove that every linear operator on X is bounded. [4]

 (b) Let $T: C[a,b]\ \rightarrow C[a,b]$ be defined by $T(f) = \Psi$, where $f\ \in C[a,b]$ and $\Psi(x) = \int_a^x f(t)\, dt,\ x \in [a,b]$. Show that T is a bounded linear operator and find its norm, where $C[a,b]$ is equipped with the sup norm $\|.\|$, given by $\|f\| =\ \sup_{t\in[a,b]} |f(t)|, f \in C[a,b]$. [5]

3. (a) Let X be a complex inner product space. Prove that $|< x,y >| \leq\ \|x\|\ \|y\|,\ \forall\ x,y\ \in X$. [4]

 (b) Let X be a complex inner product space. If a vector $x \in X$ is orthogonal to a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X and $\lim_{n\to\infty} x_n\ =\ x_0$, show that $x\perp\ x_0$. [3]

 (c) Let X be a complex inner product space and $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in X such that $\lim_{n\to\infty} x_n\ =\ x$ and $\lim_{n\to\infty} y_n\ =\ y$.
 Prove that $\lim_{n\to\infty} < x_n,\ y_n > = < x,\ y >$. [2]

4. (a) Let X be a complex inner product space and S be a nonempty subset of X. Prove that $S^{\perp\perp\perp}\ =\ S^{\perp}$. [3]

 (b) Let $(X,\ \|.\|)$ be a normed linear space and $f: X\ \rightarrow\ \mathbb{R}$ be a function defined by $f(x) =\ \|x\|,\ x\ \in X$. Prove that f is continuous. [2]

 (c) Let $A\ \subset\ \mathbb{R}^n,\ f_1,\ f_2, \dots ,\ f_m: A\ \rightarrow\ \mathbb{R}$ be m-functions and let $f: A\ \rightarrow\ \mathbb{R}^m$ be a function defined by $f(x) = (f_1(x),\ f_2(x), \dots ,\ f_m(x)),\ x \in A$. Prove that f is continuous at $a \in A$ if and only if each of the functions $f_1,\ f_2, \dots ,\ f_m$ is continuous at a. [4]

5. (a) Let $f: \mathbb{R}^n\ \rightarrow\ \mathbb{R}^m$ be a function such that f is differentiable at $c \in\ \mathbb{R}^n$. Prove that f is [4]

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continuous at c.

(b) Let $f:\mathbb{R}^3 \to \mathbb{R}^2$ be a function defined by $f(x,y,z) = (\,xyz,\ 2x+3y+5z\,)$, $\forall\,(x,y,z) \in \mathbb{R}^3$. Find $Df(1,2,3)$, where $Df(c)$ denotes the derivative of f at $c \in \mathbb{R}^3$. [5]

Group–B (Real Analysis)
[Marks: 18]

Answer any **two** questions. Only **first two** answers will be evaluated. 9× 2 = 18

1. (a) Let $f : (a,b) \to R$ be a monotonically increasing function defined on an open interval (a,b). Show that the set of points of discontinuities of f is countable. [4]

(b) Examine if the function $f:[0,1] \to R$ defined by, [5]

$$f(x) = \begin{cases} x\sin\dfrac{\pi}{x} & for\ 0 < x \leq 1 \\ 0 & for\ x = 0 \end{cases}$$

is of bounded variation.

2. (a) Let f be continuous on $[a,b]$. If the function ϕ has a bounded Riemann integrable derivative on $[a,b]$ then prove that the RS-integral $\int_a^b f(x)\,d\phi(x)$ and the Riemann integral $\int_a^b f(x)\,\phi'(x)dx$ exists and also $RS\int_a^b f(x)\,d\phi(x) = R - \int_a^b f(x)\,\phi'(x)dx$ [4]

(b) Compute the RS-integrals if exist. [5]

(i) $\int_0^\pi sinx\,d(cosx)$ (ii) $\int_0^3 (x^2+x+1)\,d[x]$

3. (a) If $\{E_n\}$ is an infinite sequence of measurable sets satisfying the condition $E_{n+1} \subset E_n$ for each n and $mE_1 < \infty$, then show that $m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} mE_n$. [5]

(b) Prove that a continuous function $f:(a,b) \to \mathbb{R}$ is measurable. [4]