M.A./M.Sc. Semester IV Examination, 2021 (CBCS)

Subject: Mathematics (Applied Stream)

Course: MMATAME406-1 (Boundary Layer Flows & Magnetohydrodynamics-II)

Time: 2 Hours | Full Marks: 40

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10 = 40$

1. Write short notes on (a) magnetic Reynolds number (b) Sausage mode of instability. [5+5]
2. State and prove Ferrero's law of isorotation. [2+8]
3. Obtain the general solution of a force free field. Define Toroidal field and Poloidal field. [6+2+2]
4. (a) Using MHD approximations, simplify the expression for total current. [4]

   (b) Prove that for a perfectly conducting fluid, lines of magnetic force are frozen in the fluid. [6]
5. (a) Derive magnetic induction equation and explain the terms involved in it. [5]

   (b) Obtain Ohm's law with Hall current for a static conductor. [5]
6. What do you mean by MHD Rayleigh problem? Dicuss the problem and set up the governing equations with appropriate boundary conditions for it. [2+2+3+3]

M.A./M.Sc. Semester IV Examination, 2021 (CBCS)

Subject: Mathematics (Applied Stream)

Course: MMATAME406-2 (Turbulent Flows-II)

Time: 2 Hours | Full Marks: 40

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10 = 40$

1. Write down properties of two fundamental correlation functions f and g. Determine the relation between f and g. [4+6]
2. (a) Establish Taylor's equation for the rate of dissipation of isotropic turbulence. [5]

   (b) Deduce Saffman invariant in turbulence. [5]

3. Define spectral functions in turbulence Let $R(r)=\frac{1}{2}R_{ij}(r)$ where $R_{ij}(r)$ is the two-point correlation function. Establish the relations, [2+4+4]

I. $R(r)=\int_0^\infty E(k)\frac{\sin(kr)}{kr}dk$,

II. $E(k)=\frac{2}{\pi}\int_0^\infty R(r)kr\sin(kr)dr$.

4. Discuss Kolmogorov equilibrium hypothesis in turbulence and hence establish Kolmogorov scaling laws on dimensional ground. [5+5]

5. Define velocity structure function of order p in turbulence. Prove Kolmogorov four-five law for homogeneous and isotropic turbulent flows. [3+7]

6. Establish Karman-Howarth equation of two correlation functions for isotropic turbulence. What is the difficulty for solving this equation? [8+2]

M.A./M.Sc. Semester IV Examination, 2021 (CBCS)

Subject: Mathematics (Applied Stream)

Course: MMATAME406-3 (Space Sciences-II)

Time: 2 Hours Full Marks: 40

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10=40$

1. Derive the (0,0)-th component of Ricci tensor for the Friedmann Metric. [10]
2. Construct the Hubble parameter for the universe for a particular dark energy model to be chosen by you. How a Hubble parameter-redshift data set can help you to constrain the free parameters of the model? [6+4]
3. Define with explanation: Scale factor, cosmological redshift, Hubble parameter, dimensionless density parameter, critical density of universe. [2×5]
4. On which parameter does Chandrasekhar’s limit depend? Explain your answer. If you incorporate rotation and charge or either of them, what will be the consequences? [4+6]
5. Construct and solve radiative transfer equation. [10]
6. Explain in brief about fine tuning problem of cosmology. [10]

M.A./M.Sc. Semester IV Examination, 2021 (CBCS)

Subject: Mathematics (Pure Stream)

Course: MMATPME406-1 (Advanced Functional Analysis-II)

Time: 2 Hours | Full Marks: 40

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10 = 40$

1. (a) Define a best approximation to a point x of a normed linear space X out of a given subspace Y of X. Find a best approximation to a point $\left(\frac{1}{3},\frac{1}{2},\frac{1}{4}\right) \in \mathbb{R}^3$ out of zx-plane. [2+2]

   (b) Let X be a normed linear space and Y be a subspace of X such that $Dim(Y) < \infty$. Prove that there exists a best approximation to $x \in X$ out of Y. Does the converse hold if $Dim(Y) = \infty$? [4+2]

2. (a) Let C be a closed convex subset of a Hilbert space H and let $x \in H$. If $\delta = \inf_{y\in C} \|x - y\|$, then show that there exists a unique element $y_0 \in C$ such that $\|x - y_0\| = \delta$. Hence show that there exists a unique element in C having smallest norm. [5+2]

   (b) Examine whether $C(X, \mathbb{R})$ is a lattice. [3]

3. Define an ideal in a Banach algebra X with identity e. Let M be a maximal ideal in a commutative Banach algebra X with identity e. Then prove the following: [2+2+4+2]
   (i) M is closed.
   (ii) If I be a proper ideal of X then there exists a maximal ideal in X such that $I \subseteq M$.
   (iii) If $x \in I$, then x is not invertible, I being a proper ideal of X.

4. (a) Define weak topology on a normed linear space. Prove that the notion of weak convergence of a sequence of elements in a normed linear space X coincides with the notion of convergence with respect to weak topology on X. [2+3]

   (b) Prove that a subspace of a normed linear space is weakly closed if and only if it is strongly closed. [5]

5. Define (i) spectrum and (ii) spectral radius of an element in a Banach algebra with identity e. Prove that $\sigma(x)$, the spectrum of x, is compact, assuming $\sigma(x)$ non-empty. [2+2+6]

6. (a) When is a normed linear space said to be weakly complete? If a normed linear space X is reflexive, then prove that it is weakly complete. [2+5]

   (b) Prove that the collection of all non-invertible elements in a Banach algebra X with identity e forms a closed set in X. [3]

M.A./M.Sc. Semester IV Examination, 2021 (CBCS)
Subject: Mathematics (Pure Stream)
Course: MMATPME406-4 (Rings of Continuous Function II)

Time: 2 Hours Full Marks: 40

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10 = 40$

1 (a) If I is a z-ideal of C, then show that C/I is totally ordered if and only if I is prime. [2+2]

(b) Show that any prime z-filter with countable intersection property is contained in unique real-z-ultrafilter. [4]

(c) Show that arbitrary intersection of absolutely convex ideals of $C(X)$ is absolutely convex. [2]

2 (a) Let $\omega_1$be the first uncountable ordinal space. Show that of any two disjoint closed sets in ω_1, at least one is countable. [4]

(b) Show that $C(\omega_1)$ contains exactly one real maximal ideal which is not fixed and contains no hyper-real maximal ideal. [4+2]

3 (a) Show that a space is pseudocompact if and only if every non-empty zero set in βX meets X. [3+3]

(b) Show that βX is extremally disconnected if and only if X is extremally disconnected. [2+2]

4 (a) Show that for any two positive integers n and m, $f^{2n} + g^{2m} \in M^p$ if and only if $\sigma f + \delta g \in M^p$, for any $\sigma, \delta \in C(X)$. [4]

(b) Show that for any maximal ideal M of $C(X)$, $M \cap C^*(X)$ is a maximal ideal of $C^*(X)$ if and only if M is real maximal ideal of $C(X)$. [3+3]

5 (a) Show that for any $p \in \beta X$, $M^p = \{f \in C(X): (fg)^*(p) = 0, \text{for every } g \in C(X)\}$. [6]

(b) Show that an arbitrary intersection of realcompact subsets is realcompact. [4]

6 (a) Show that a closed subset of a realcompact space is realcompact. [3]

(b) Show that any non-empty $G_\delta -$ set in υX must intersect X. [4]

(c) Show that $\upsilon_f X = \{p \in \beta X: f^*(p) \in \mathbb{R}\}, f \in C(X)$ is a realcompact subset of βX. [3]

M.A./M.Sc. Semester IV Examination, 2021 (CBCS)
Subject: Mathematics (Pure Stream)
Course: MMATPME406-5 (Advanced Complex Analysis-II)

Time: 2 Hours Full Marks: 40

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10 = 40$

1. (a) Let $u(x,y)$ be a harmonic function in a domain D with harmonic conjugate $v(x,y)$, and let z_0 be an arbitrary point of D and $\Delta = \Delta(z_0)$ be the distance between z_0 and the boundary of D. Show that $u(x,y)$ and $v(x,y)$ can be represented in the Poisson's integral forms. [7]

   (b) Let D be open connected set in $\mathbb{C}$, and let $u_1(x,y)$, $u_2(x,y)$ be two functions harmonic on D and continuous on $\overline{D}$. If $u_1(x,y) = u_2(x,y)$ on the boundary of D, then show that $u_1(x,y) = u_2(x,y)$ on D. [3]

2. (a) Prove that any harmonic function defined in a domain D has the mean value property. [5]

   (b) Give a physical interpretation of the Dirichlet problem for a disk. [5]

3. Prove that the set of period points of a non-constant periodic function cannot have any finite limit point. [10]

4. (a) What is a doubly periodic function? [2]

   (b) Prove that every doubly periodic entire function is a constant. [8]

5. (a) Let $n(x)$ denote the numbers of zeros of $f(z)$ in $|z| < x$, where a zero is counted according to its multiplicity. If $r_m < R \le r_{m+1}$, then show that $\int_0^R \frac{n(x)}{x}dx = \frac{1}{2\pi}\int_0^{2\pi} log\left|f\left(Re^{i\phi}\right)\right| d\phi - log|f(0)|$, where $f(0) \neq 0$. [5]

   (b) Let f be regular and $|f(z)| \le M$ in $|z| \le R$ and suppose that $f(0) \neq 0,\infty$. Show that the number of zeros of f in the disk $|z| \le \delta R (0 < \delta < 1)$ does not exceed $\frac{1}{log\,\delta} log\frac{M}{|f(0)|}$. [5]

6. (a) Show that for any complex number a, $\frac{1}{2\pi}\int_0^{2\pi} log\left|a - e^{i\theta}\right| d\theta = log^+|a|$. [4]

   (b) For a meromorphic function $f(z)$, prove that $\frac{1}{2\pi}\int_0^{2\pi} m\left(r, e^{i\theta}; f\right)d\theta \le log\,2$. [6]

M.A./M.Sc. Semester IV Examination, 2021 (CBCS)
Subject: Mathematics (Pure Stream)
Course: MMATPME406-6 (Measure and Integration – II)

Time: 2 Hours Full Marks: 40

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Notation and symbols have their usual meaning]

Answer any **four** questions. Only **first four** answers will be evaluated. $4\times 10 = 40$

1. (a) Let μ be a signed measure on a measurable space $(X,\ S)$. If $\{E_n\}_{n\in\mathbb{N}}$ is a monotone [3]

increasing sequence in S, then prove that $\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$.

(b) State and prove Hahn-decomposition theorem. [2+5]

2. (a) Let μ be a signed measure on a measurable space $(X,\ S)$ and $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. If μ is finite, then show that the measures $\mu^+,\ \mu^-$ are also finite. [3]

(b) Let μ be a signed measure on a measurable space $(X,\ S)$. Prove that $f \in L_1^r(X, S, \mu)$ if and only if $f \in L_1^r(X, S, \mu^+) \cap L_1^r(X, S, \mu^-)$. [4]

(c) Define a complex measure. Give an example of a complex measure. [2+1]

3. (a) Let ρ be a complex measure on a measurable space $(X,\ S)$. Prove that there exist finite measures $\rho_1,\ \rho_2,\ \rho_3,\ \rho_4$ on (X, S) such that [7]

   (i) $\rho_1 \perp \rho_{2,}\rho_3 \perp \rho_4$ and

   (ii) $\rho(E) = \rho_1(E) - \rho_2(E) + i\,\rho_3(E) - i\,\rho_4(E),\ \forall\, E \in S.$

(b) Let (X, S, μ) and (Y, T, ρ) be two finite measure spaces. For each $V \in S \times T$, let $f_V : X \to \mathbb{R}^*, g^V : Y \to \mathbb{R}^*$ be defined by $f_V(x) = \rho(V_x),\ \forall\, x \in X$ and $g^V(y) = \mu(V^y),\ \forall\, y \in Y$. If $A \times B$ is a measurable rectangle of $X \times Y$, then show that $f_{A\times B}$ is S-measurable and $g^{A\times B}$ is T-measurable.(Here$\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$). [3]

4. (a) Let (X, S, μ) and (Y, T, ρ) be two finite measure spaces. Define product measure $\mu \times \rho$ on the measurable space $(X \times Y,\ S \times T)$. Show that $\mu \times \rho$ is a measure $S \times T$. Further if $E \in S$ and $F \in T$, then show that $(\mu \times \rho)(E \times F) = \mu(E)\rho(F)$. [1+4+2]

(b) Let (X, S, μ) be a measure space and f be a measurable function on X. Show that $|f| \le ess\ \sup|f|\ \ a.e.$ on X. [3]

5. (a) Let $f \in L^p[a,b],\ 1 \le p < \infty$ and $\epsilon > 0$. Prove that there exists a simple function $s \in L^p[a,b]$such that $\|f - s\|_p < \epsilon$ and $|s| \le |f|$. [7]

(b) Let μ be a signed measure on a measurable space $(X,\ S)$. If $E, F \in S, E \subset F$ and $|\mu(F)| < \infty$, then show that $|\mu(E)| < \infty$ and $\mu(F - E) = \mu(F) - \mu(E)$. [3]

6. (a) Let $f \in L^p[a,b]$ and $g \in L^q[a,b]$, where $1 < p < \infty, 1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $fg \in L^1[a,b]$ and $\int_a^b |fg|\, dm \le \|f\|_p \|g\|_q$, where m denotes Lebesgue measure. [6]

(b) Let (X, S) and (Y, T) be measurable spaces and $\hat{E}$ be the class of all elementary sets in $X \times Y$. Prove that $\hat{E}$ is an algebra. [4]