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M.A./M.Sc. Semester IV Examination, 2020 (Old Pattern under CDOE)

Subject: Mathematics (Applied Stream)

Time: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[*Notation and symbols have their usual meanings*]

Paper: MAT 405 (**Advanced Operations Research**)

Answer any **five** questions. Only the **first five** answers will be evaluated. 5×10 = 50

1. (a) Find the maximum of the function $f(x) = 4x_1 + x_2 + 15$ subject to $x_1 + 4x_2^2 = 5$, using Lagrange multipliers method. Also, find the effect of changes of the right hand side of the constraint on the optimal value of *f*. [6+4]

2. Apply Wolfe's method for solving the following quadratic programming problem:

   Maximize $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

   subject to $x_1 + 2x_2 \le 2$ and $x_1, x_2 \ge 0.$ [10]

3. (a) Use the Kuhn-Tucker necessary conditions to solve the following optimization problem:

   Maximize $z = 2x_1 - x_1^2 + x_2$

   subject to $2x_1 + 3x_2 \le 6$, $2x_1 + x_2 \le 4$ and $3x_1 + 9x_2 = 16.$ [6]

   (b) Using Golden section method,

   Maximize $f(x) = \begin{cases} \dfrac{2x+3}{6} & x \le 3 \\ 6 - x & x > 3 \end{cases}$

   in the interval [-1,5] (use 5 experiments). [4]

4. (a) For the quadratic function $f(x) = \frac{1}{2}\langle Ax, x\rangle$ (*A* being positive definite matrix), show that the gradient vector $\left\{g^{(k)}\right\}$ are mutually orthogonal and the direction search vectors $\left\{d^{(k)}\right\}$ are mutually *A*-conjugate. [5]

   (b) Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from $x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ using conjugate gradient method. [5]

5. (a) Derive the steady state equations for the queueing model $(M/M/1):(N/FCFS/\infty)$. [6]

   (b) In a car wash service facility information gather indicates that cars arrive for service according to a Poisson distribution with mean five per hour. The time for washing and cleaning for each car varies but is found to follow an exponential distribution with mean 10 minutes per car. The facility cannot handle more than one car at a time and has a total of 5 [2+2]

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parking spaces. If the parking spot is full, newly arriving cars balk to seek services elsewhere.

(i) How many customers the manager of the facility is losing due to the limited parking space?

(ii) What is the expected waiting time until a car is washed?

6. (a) Write down the dual of the primal with equality type constraints by geometric programming technique. [2]

(b) Solve the following optimization problem by geometric programming technique: [8]

Minimize $f(x) = 7x_1x_2^{-1} + 3x_2x_3^{-2} + 5x_1^{-3}x_2x_3 + x_1x_2x_3$

7. (a) State Bellman's principle of optimality. [2]

(b) A student has to take examination in three courses *X*, *Y*, *Z*. He has three days available for study. He feels that it would be better to devote a whole day to the study of the same course, so that he may study a course for one day, two days or three days or not at all. His estimates of grades, he may get by the study, are as follows: [8]

Study days \ Course	*X*	*Y*	*Z*
0	1	2	1
1	2	2	2
2	2	4	4
3	4	5	4

How should he plan to study so that he maximizes the sum of his grades? Solve by Dynamic Programming technique.

Paper: MAT 405 (Viscous Flows, Boundary Layer Theory and MHD)

Answer any **five** questions. Only the **first five** answers will be evaluated. 5×10 = 50

1. Discuss 'slow motion' of viscous incompressible fluid. Write practical applications for this kind of motion. Find mathematical equation for slow motion. [3+3+4]

2. Write constitutive equations for MHD. Establish Lorentz force per unit volume of a conducting material. Simplify it using MHD approximations. [3+4+3]

3. (a) Explain, physically, the displacement and energy thicknesses in boundary layer flow over a plate. [5]

(b) Discuss flow separation in steady boundary layer flows. [5]

4. Discuss each term of the magnetic induction equation physically. Approximate the equation for flows of conducting fluid with high and low magnetic Reynolds numbers and explain physically. [5+5]

5. Write short notes (any two) on: [5+5]

(a) Significance of Reynolds number for the viscous fluid motion.

(b) Self-similar flows.

(c) Iso-rotation and Ferraro's law.

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(d) Alfven waves.

6. Discuss MHD Couette steady flow between two non – conducting parallel plates in the presence of a uniform transverse magnetic field.Determine the expression for velocity field. [5+5]

7. What do you mean by linear pinch effect ? Discuss theta pinch configuration. [4+6]

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Paper: MAT 405 (Differential Geometry of Manifolds)

Answer any **five** questions. Only the **first five** answers will be evaluated. 5×10 = 50

1. (a) Define the tangent space at a point of a smooth manifold. Find the tangent space at the point $p=(1,1,1)$ to the surface S of $\mathbb{R}^3$ defined by the equation $x^3-y^3+xyz-xy=0$. [2+5]

   (b) Show that every smooth manifold is locally connected. [3]

2. (a) When is a vector field on a smooth manifold said to be complete? Is every vector field on the real line complete? Justify your answer. [1+3]

   (b) If X is a smooth vector field on a smooth compact manifold Mthen show that X determines a 1-parameter group of transformations on M. [6]

3. (a) Let $f: M \to N$ be a smooth map from a smooth manifold M to a smooth manifold N. Show that the pull-back map f^* of f commutes with the exterior derivative d for any r-form on N. [6]

   (b) Show that the vector space $\mathbb{R}^3$ with the operation cross product of vectors is a Lie algebra. [4]

4. Write short notes on (i) linear frame bundle (ii) associated principal bundle and (iii) bundle homomorphism. [3+4+3]

5. (a) Define Lie transformation group that acts on a smooth manifold M from the right. [2]

   (b) Let G be a Lie transformation group acting on a smooth manifold M from the right. Show that the set of all fundamental tangent vector fields on M forms a Lie algebra which is homeomorphic to the Lie algebra T_eG of G, where e is the identity element of G. [8]

6. (a) Define Nijenhuis tensor on an almost complex manifold. Show that it is pure in two [2+2]

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slots.

(b) If the F-connection on an almost complex manifold is symmetric then prove that the Nijenhuis tensor vanishes identically. [6]

7. (a) Obtain the relation between second fundamental form and shape operator on the submanifold of a Riemannian manifold. [3]

(b) Deduce Ricci equation for the submanifold of a Riemannian manifold. [7]

Paper: MAT 405 (**Advanced Functional Analysis**)

Answer any **five** questions. Only the **first five** answers will be evaluated. 5×10 = 50

1. (a) Define a local base in a topological vector space. If B is a local base in a topological vector space X then show that every member of B contains the closure of some member of B. [2+2]

(b) Let X be a vector space and $A \subset X$ be convex and absorbing. Define Minkowski functional of A. Let μ_A is the Minkowski functional of A. If $B = \{x \in X : \mu_A(x) < 1\}$ and $C = \{x \in X : \mu_A(x) \le 1\}$ then show that $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$. [1+3+2]

2. Define a locally convex topological vector space? Give an example of it with justifications. Examine if sequence space $l_{\frac{1}{2}}$,where $l_{\frac{1}{2}} = \left\{ \{x_i\} \in \mathbb{C} : \sum_i |x_i|^{\frac{1}{2}} < \infty \right\}$ is a locally convex topological vector space. [2+2 +6]

3. (a) Let X be a normed linear space and let X^* be the first conjugate space of X. Define weak* convergence of a sequence of elements in X^*. If $\{x_n^*\}_{n=1,2,...}$ be a sequence of elements in X^* such that $\{x_n^*\}$ is weak* convergent to an element of $x^* \in X'$. Prove that $\|x^*\| \le \lim_{n\to\infty} \inf \|x_n^*\|$. [2+2]

(b) If X is a reflexive Banach space then show that every bounded sequence in X has a weakly convergent subsequence. [6]

4. (a) If ϕ is a multiplicative linear functional on a complex Banach algebra X with an identity e then show that $\phi(e) = 1$ and $\phi(x) \ne 0$ for every invertible elements $x \in X$. [4]

(b) Suppose X be a complex Banach algebra with an identity e. If $x \in X$ with $\|x\| < 1$ then show that [6]

(i) $(e - x)$ is invertible ,

(ii) $\left\|(e-x)^{-1} - e - x\right\| \le \dfrac{\|x\|^2}{1 - \|x\|}$, and

(iii) $|\phi(x)| < 1$, for every multiplicative linear functional ϕ on X

5. When is a set said to be an extreme set in a vector space? Suppose X is a topological [2+8]

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vector space on which X^*, the fist conjugate space of X, separates points. If K is a compact convex set in X then show that K is the closed convex hull of the set of extreme points in K.

6. Define an involution on a complex Banach algebra X. Suppose X be a complex Banach algebra with an involution. For $x \in X$ suppose that $\sigma(x)$ is the spectrum of x. If x^* is the conjugate of x and e is the identity element in X then prove the following:: [2×5]

(i) $x + x^*, i(x - x^*)$, and xx^* are self adjoint,

(ii) x has a unique representation $x = u + v$, where $u, v \in X$ are self adjoint,

(iii) e is self adjoint ,

(iv) x is invertible if and only if x^* is invertible in which $\left(x^*\right)^{-1} = \left(x^{-1}\right)^*$,and

(v) $\lambda \in \sigma(x)$ if and only if $\lambda^* \in \sigma\left(x^*\right)$

7. When is a normed linear space said to be (i) strictly convex and (ii) uniformly convex? Let $C[0,1]$ be the space of all real valued continuous functions over [0, 1] with the sup-norm $\|x\|_0 = \sup\limits_{0 \le t \le 1} |x(t)|$, where $x \in C[0,1]$. For a fixed $\mu > 0$, define [2+(4+4)]

$\|x\|_\mu = \|x\|_0 + \mu \left(\int\limits_0^1 x^2(t)\, dt \right)^{\frac{1}{2}}$. Show that $\|.\|_0$ and $\|.\|_\mu$ are equivalent norms in $C[0,1]$.

Moreover show that $\left(C[0,1], \|.\|_\mu\right)$ is a strictly convex normed linear space, while it is not uniformly convex.

Paper: MAT 405 (**Operator Theory and Applications**)

Answer any **five** questions. Only the **first five** answers will be evaluated. 5×10 = 50

1. Let X be a normed linear space with dualX'. Define annihilator of a subset of X and X'.Show that annihilator of a subset and the orthogonal complement of that subset coincide in a Hilbert space. If $T: X \to Y$ is a bounded linear transformations and T'be itsconjugate, where $X$and Y are normed linear spaces, then show that (i)$\overline{R(T)}^a = N(T')$. (ii) $\overline{R(T)} = a_{N(T')}$(iii) $a_{\overline{R(T')}} = N(T)$ [2+2+6]

2. Let X be a complex Hilbert space and let $\{A_n\}$ be a sequence of self-adjoint, commuting transformations from $B(X, X)$. Let $B \in B(X, X)$ be a self-adjoint transformation such that $A_j B = B A_j$ for all j and further suppose that $A_1 \le A_2 \le \cdots. \le A_n \le \cdots. \le B$. Prove that there exists a self-adjoint bounded linear transformation A such that $A_n \to A$ strongly and $A \le B$. [10]

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3. If A is a compact operator defined on a normed linear space X into a normed linear space Y, then show that (i) $\alpha A$is compact where α is a scalar and (ii) A^* is compact where A^* denotes the adjoint of A. [3+7]

4. Let X and Y be normed linear spaces and $T \in B(X,Y)$. Define the spectrum $\sigma(T)$ of T. If $T \in B(X,X)$ where X is a Banach space then prove that $\sigma(T)$ is compact. [2+8]

5. Let X be a complex Hilbert space and let $T \in B(X,X)$ be compact normal but not finite dimensional. Suppose that $\sigma_p(T) = \{\lambda_1, \lambda_2, \dots \dots . \lambda_n \dots \dots .\}$ such that $|\lambda_1| \geq |\lambda_2| \dots \dots . . |\lambda_n| \geq \cdots$ and $\{\lambda_n\}$ converges to zero. Then prove that $\{T_n\}$ converges uniformly on X where $T_n = \sum_{i=1}^{n} \lambda_i E_i$, E_i being the orthogonal projection on $N(T - \lambda_i I)$, $i = 1,2, \dots .$ [10]

6. Let X be a complex Hilbert space and let $T \in B(X,X)$ be self-adjoint. Then prove that (i) a complex number λ belongs to $\rho(T)$ if and only if there exist a real number $c > 0$ such that $\|T_\lambda(x)\| \geq c\|x\|$ for every $x \in X$ and (ii) the residual spectrum of T is empty. [7+3]

7. Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H $(\neq \{\theta\})$. Also let E_λ (λ real) be the projection of H onto the null space $N(T_\lambda^+)$ of the positive part T_λ^+ of $T_\lambda = T - \lambda I$. Then prove that $\xi = (E_\lambda)_{\lambda \in R}$ is a spectral family on the interval $[m, M]$where $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ and M= $\sup_{\|x\|=1} \langle Tx, x \rangle$. [10]